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Approximations for Chromatic Polynomials

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A sequence of finite graphs may be constructed from a given graph by a process of repeated amalgamation. Associated with such a sequence is a transfer matrix whose minimum polynomial gives a recursion for the chromatic polynomials of the graphs in the sequence. Taking the limit, a generalised "chromatic polynomial" for infinite graphs is obtained.

1. INTRODUCTION

This paper is motivated by the problem of defining a "chromatic polynomial" for infinite graphs. The theoretical problem is apparently a difficult one, and so it may be helpful to begin in a fairly straightforward way.

For some infinite graphs Ψ it is possible to find a sequence $\{\Psi_s\}$ of finite subgraphs whose limit is Ψ. (All terms used without explanation in this introduction must be given their intuitive meanings; their precise definitions are part of our fundamental problem.) Further, if $C(\Psi_s; u)$ denotes the chromatic polynomial of Ψ_s in the variable u, then in certain circumstances the limit

$$C_\infty(\Psi; u) = \lim_{s\to\infty} \{C(\Psi_s; u)\}^{1/|V\Psi_s|}$$

exists for a range of values of u. This limit is a good candidate for the "chromatic polynomial" of Ψ, although of course it is not a polynomial. Clearly, there are major problems of existence and uniqueness associated with the definition of C_∞, but it is not our purpose to investigate such questions here.

Our aim is to investigate a few simple cases in which it is possible to find approximations for C_∞. It is not hard to see that if $C_\infty(\Psi; u)$ exists then the infinite graph Ψ must be countable. Some measure of regularity is also desirable, and so we deal only with vertex-transitive graphs. Lastly, the graphs discussed will be planar, because of their relevance to problems of physics.

In the course of our exposition, we shall derive several new results which could be useful in the theory of chromatic polynomials of finite graphs.

2. The Chromatic Polynomial of an Amalgamated Graph

We shall develop a recursive technique for the calculation of chromatic polynomials. This technique has previously been used in an investigation [3] of the Tutte polynomials of certain graphs, and its formalization was initiated by D. A. Sands in his thesis [6]. In this paper we shall give a theoretical treatment of the method as it may be applied to chromatic polynomials; the generalization to Tutte polynomials is possible, but it would involve rather a lot of cumbersome notation.

If u is a positive integer, we use the term *u-colouring* of a graph Γ to denote a function from the vertex-set $V\Gamma$ to the set

$$[u] = \{1, 2, 3, \ldots, u\}$$

with the property that adjacent vertices are given different values. The number of u-colourings of Γ is the value $C(\Gamma; u)$ of the *chromatic polynomial* $C(\Gamma)$. All graphs considered may have loops, in which case $C(\Gamma) = 0$, or they may have multiple edges, in which case $C(\Gamma) = C(\bar{\Gamma})$, where $\bar{\Gamma}$ is derived from Γ by deleting all but one of each set of multiple edges.

Let J be any given subset of $V\Gamma$. The set of all equivalence relations on J is partially ordered by the rule

$$\alpha \geqslant \beta \Leftrightarrow (x, y) \in \beta \Rightarrow (x, y) \in \alpha;$$

so that $\alpha \geqslant \beta$ whenever the α-classes are unions of β-classes. Any function f defined on J has an associated equivalence relation $\pi(f)$ on J, given by

$$(x, y) \in \pi(f) \Leftrightarrow f(x) = f(y).$$

We shall say that f is of *type* $\pi(f)$, and that f is *compatible* with the equivalence relation α if $\pi(f) \geqslant \alpha$.

For the time being let us fix Γ, J, and u. We define three functions $P(\alpha)$, $Q(\alpha)$, $R(\alpha)$ as follows. $P(\alpha)$ is the number of u-colourings f of Γ for which $\pi(f \mid J) = \alpha$, $Q(\alpha)$ is the number of u-colourings of Γ which are compatible with α (that is, such that $\pi(f \mid J) \geqslant \alpha$), and $R(\alpha)$ is the number of ways of extending a given function $g: J \to [u]$, of type α, to a colouring of Γ. It is clear that $R(\alpha)$ depends only on α, not on g.

The functions P, Q, R are related by some simple identities. From the obvious relation

$$Q(\alpha) = \sum_{\beta \geqslant \alpha} P(\beta) \tag{2.1}$$

we derive by Möbius inversion

$$P(\alpha) = \sum_{\beta \geqslant \alpha} \mu(\beta, \alpha)\, Q(\beta), \tag{2.2}$$

where μ is the Möbius function for the lattice of equivalence relations (partitions) given explicitly by Rota [5]. The relationship between P and R is equally simple: if α has r equivalence classes on J, then u colours can be assigned to these classes in $\phi(\alpha) = u(u-1)(u-2) \cdots (u-r+1)$ ways so that each class has a different colour. Thus

$$P(\alpha) = \phi(\alpha)\, R(\alpha). \tag{2.3}$$

LEMMA 1. *$P(\alpha)$, $Q(\alpha)$, and $R(\alpha)$ are polynomial functions of u. Their degrees are $|V\Gamma| - |J| + r$, $|V\Gamma| - |J| + r$ and $|V\Gamma| - |J|$, respectively.*

Proof. Let Γ^α denote the graph obtained from Γ by identifying each set of α-equivalent vertices. Then Γ^α has $|J| - r$ vertices fewer than Γ, and the same number of edges as Γ. There is a (1, 1)-correspondence between the u-colourings of Γ^α and the u-colourings of Γ which are compatible with α, so

$$Q(\alpha) = C(\Gamma^\alpha)$$

showing that $Q(\alpha)$ is a polynomial in u with degree $|V\Gamma| - |J| + r$.

Similarly, $P(\alpha) = C(\Gamma_\alpha)$, where Γ_α is derived from Γ^α by adding edges joining each pair of vertices resulting from the identifications on J. Thus $P(\alpha)$ is a polynomial in u with the same degree as $Q(\alpha)$.

Finally, since Γ_α contains a complete graph on r vertices, its chromatic polynomial $C(\Gamma_\alpha\,;u)$ has a factor $\phi(\alpha)$ and from (2.3) it follows that $R(\alpha)$ is the complementary factor, which has degree $|V\Gamma| - |J|$. ∎

We now apply these ideas to obtain a formula for the chromatic polynomial of an amalgamated graph. If Γ and Δ are graphs with $V\Gamma \cap V\Delta = J$ and whose common edges are incident only with vertices in J, then we denote by $\Gamma * \Delta$ the graph whose vertices and edges are those of Γ and those of Δ. Since a function $g: J \to [u]$ may be extended independently to the rest of Γ and the rest of Δ, we have

$$R(\Gamma * \Delta, \alpha) = R(\Gamma, \alpha)\, R(\Delta, \alpha).$$

Here, and in what follows, we modify our previous notation to display the graphs under consideration.

LEMMA 2.
$$C(\Gamma * \Delta) = \sum_{\alpha} P(\Gamma, \alpha)\, R(\Delta, \alpha),$$

where the sum is over all equivalence relations α on J.

Proof.
$$\begin{aligned} C(\Gamma * \Delta) &= \sum_{\alpha} P(\Gamma * \Delta, \alpha) \\ &= \sum_{\alpha} R(\Gamma * \Delta, \alpha)\, \phi(\alpha) \\ &= \sum_{\alpha} R(\Gamma, \alpha)\, R(\Delta, \alpha)\, \phi(\alpha) \\ &= \sum_{\alpha} P(\Gamma, \alpha)\, R(\Delta, \alpha). \quad \blacksquare \end{aligned}$$

The result of Lemma 2 can be written as

$$C(\Gamma * \Delta) = \sum_{\alpha} R(\Delta, \alpha)\, C(\Gamma_{\alpha}), \tag{2.4}$$

and in this form we interpret it as follows. The chromatic polynomial of the graph obtained by attaching Δ to Γ is a linear combination of the chromatic polynomials of the graphs Γ_{α}, with coefficients which depend only on Δ. In practice, it is more convenient to use the graphs Γ^{α}, and to do this we need (2.2), which expresses $C(\Gamma_{\alpha}) = P(\alpha)$ in terms of $C(\Gamma^{\alpha}) = Q(\alpha)$. Substituting in (2.4) we find

$$C(\Gamma * \Delta) = \sum_{\alpha} R(\Delta, \alpha) \sum_{\beta \geqslant \alpha} \mu(\beta, \alpha)\, C(\Gamma^{\beta}).$$

THEOREM 1. *The chromatic polynomial of an amalgamated graph $\Gamma * \Delta$ is a linear combination of the chromatic polynomials of the graphs Γ^{β}, with coefficients depending only on Δ.*

Proof. Interchanging the order of the double sum above we get

$$C(\Gamma * \Delta) = \sum_{\beta} W(\Delta, \beta)\, C(\Gamma^{\beta}), \tag{2.5}$$

where the coefficients $W(\Delta, \beta)$ are given by

$$W(\Delta, \beta) = \sum_{\alpha \leqslant \beta} \mu(\beta, \alpha)\, R(\Delta, \alpha). \quad \blacksquare$$

We notice that, if ι denotes the identical relation, then

$$W(\Delta, \iota) = R(\Delta, \iota). \tag{2.6}$$

The formula (2.5) is a generalization of the well-known formula [2, p. 61] for the case when J spans a complete graph in Γ and Δ. In that case Γ^β contains a loop except when $\beta = \iota$, and $\Gamma^\iota = \Gamma$, so the only nonzero term in (2.5) is $W(\Delta, \iota)\, C(\Gamma)$. But

$$W(\Delta, \iota) = R(\Delta, \iota) = \frac{P(\Delta, \iota)}{\phi(\iota)} = \frac{C(\Delta_\iota)}{\phi(\iota)};$$

also $C(\Delta_\iota) = C(\Delta)$ and $\phi(\iota)$ is the chromatic polynomial of the complete graph K on the vertex-set J. Thus we obtain the aforementioned formula

$$C(\Gamma * \Delta) = \frac{C(\Gamma)\, C(\Delta)}{C(K)}. \tag{2.7}$$

3. The Transfer Matrix

In this section we shall investigate the chromatic polynomials of graphs formed by repeated amalgamation.

Let Λ be a graph in which there are two subsets J_1 and J_2 of $V\Lambda$ with a given (1, 1)-correspondence between them. There will be no loss of generality in supposing that J_1 is an independent set of vertices, and we shall make this assumption henceforth. We construct a sequence $\{\Gamma_j\}$ of graphs by taking Γ_0 to be a given graph with $J_2 \subseteq V\Gamma_0$, and defining

$$\Gamma_j = \Gamma_{j-1} * \Lambda \qquad (j \geqslant 1).$$

Here the $*$ operation is to be interpreted as meaning that the subset J_1 of $V\Lambda$ is identified with corresponding set J_2 added at the previous stage.

For example, if Λ is the path graph P_4 with vertices 1, 2, 3, 4 in linear order, $J_1 = \{1, 4\}$, $J_2 = \{2, 3\}$ and Γ_0 is a path 23, then Γ_j is just a chain of j squares. In this example the relationship between $C(\Gamma_j)$ and $C(\Gamma_{j-1})$ is easily obtained from (2.7):

$$C(\Gamma_j\,;u) = (u^2 - 3u + 3)\, C(\Gamma_{j-1}\,;u).$$

In general, the chromatic polynomial $C(\Gamma_j)$ will be a linear combination of the chromatic polynomials $C(\Gamma_{j-1}^\beta)$ with coefficients depending only on Λ, as in Theorem 1. Further, if Γ_j^α is regarded as the amalgamation of Γ_{j-1} and Λ^α, then $C(\Gamma_j^\alpha)$ is also a linear combination:

$$C(\Gamma_j^\alpha) = \sum_\beta W(\Lambda^\alpha, \beta)\, C(\Gamma_{j-1}^\beta). \tag{3.1}$$

The expression for $C(\Gamma_j)$ is the special case $\alpha = \iota$.

It is convenient to express (3.1) in matrix form. We introduce the column vector $\mathbf{q}_j$, and the *transfer matrix* T whose rows and columns are labeled by the equivalence relations α on J_1, and whose entries are given by

$$(\mathbf{q}_j)_\alpha = C(\Gamma_j^\alpha), \qquad t_{\alpha\beta} = W(\Lambda^\alpha, \beta).$$

Then the equation (3.1) becomes a matrix equation over the ring $\mathbb{Z}[u]$ of integer polynomials in u:

$$\mathbf{q}_j = T\mathbf{q}_{j-1}. \tag{3.2}$$

THEOREM 2. *Let T be the transfer matrix associated with a family $\{\Gamma_j\}$ and suppose that the minimum polynomial of T is*

$$m(x) = x^\rho + a_1 x^{\rho-1} + \cdots + a_\rho.$$

Then the chromatic polynomial $C(\Gamma_j)$ satisfies the linear recursion

$$C(\Gamma_{j+\rho}) + a_1 C(\Gamma_{j+\rho-1}) + \cdots + a_\rho C(\Gamma_j) = 0,$$

where the coefficients belong to $\mathbb{Z}[u]$, and are independent of j.

Proof. Since $m(T)$ is the zero matrix we get

$$m(T)\,\mathbf{q}_j = \mathbf{q}_{j+\rho} + a_1\mathbf{q}_{j+\rho-1} + \cdots + a_\rho\mathbf{q}_j = \mathbf{0}.$$

Taking the component of this equation corresponding to $\alpha = \iota$ we obtain the desired result. ∎

In the terminology of [3], we have shown that Γ_j is a *recursive family* of graphs. The standard methods for the solution of a linear recursion can thus be used to obtain (in principle) an explicit formula for $C(\Gamma_j)$.

In practice, several of the graphs Γ_j^α may have loops. These can be disregarded, since their chromatic polynomials vanish identically. As an example, take Λ to be the graph of Fig. 1, with $J_i = \{a_i, b_i, c_i\}$ $(i = 1, 2)$, and Γ_0 to be a path $a_2b_2c_2$.

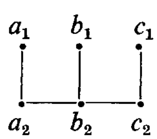


FIGURE 1.

The resulting graph Γ_j is a double strip of $2j$ squares. There are five partitions of J_1, but only ι and $\theta = \{\{a_1, c_1\}, \{b_1\}\}$ give loopless graphs. The equation (3.2) for this family is:

$$\begin{bmatrix} C[\Gamma_j ; u) \\ C(\Gamma_j^{\theta}; u) \end{bmatrix} = \begin{bmatrix} u^3 - 5u^2 + 10u - 8 & 1 \\ u^2 - 4u + 5 & u - 2 \end{bmatrix} \begin{bmatrix} C(\Gamma_{j-1} ; u) \\ C(\Gamma_{j-1}^{\theta} ; u) \end{bmatrix}.$$

In a simple example like this one the entries of the transfer matrix are most easily found by applying the standard "deletion and contraction" method to the graphs Γ_j^{α}. The technique is to delete and contract the edges of Λ until $C(\Gamma_j^{\alpha})$ is expressed in terms of the chromatic polynomials $C(\Gamma_{j-1}^{\beta})$.

4. A Generalized Birkhoff–Whitney Expansion

The entries of the transfer matrix are W functions, defined in terms of the basic R functions. We now derive an explicit expression for $R(\Gamma, \alpha)$ in terms of the edge-subgraphs of Γ. The method used is closely related to that employed in [2, p. 65] to obtain the Birkhoff–Whitney expansion of the chromatic polynomial.

For each function f: $V\Gamma \to [u]$, not necessarily a u-colouring, define an indicator function $\hat{f}$: $E\Gamma \to \{0, 1\}$ by the rule

$$\hat{f}(e) = \begin{cases} 1 & \text{if } e \text{ is incident with two vertices } x \text{ and } y, \text{ and } f(x) \neq f(y); \\ 0 & \text{otherwise.} \end{cases}$$

Let J be a subset of $V\Gamma$, α an equivalence relation on J, and g: $J \to [u]$ a given function of type α. Let $M(\Gamma)$ denote the set of all functions

$$f\colon V\Gamma \to [u]$$

such that f coincides with g on J. From the definition of $R(\Gamma, \alpha)$ we have

$$\begin{aligned} R(\Gamma, \alpha) &= \sum_{f \in M(\Gamma)} \prod_{e \in E\Gamma} \hat{f}(e) \\ &= \sum_{f \in M(\Gamma)} \prod_{e \in E\Gamma} (\{\hat{f}(e) - 1\} + 1) \\ &= \sum_{f \in M(\Gamma)} \sum_{S \subseteq E\Gamma} \prod_{e \in S} (\hat{f}(e) - 1). \end{aligned}$$

Consider the double sum. For a given subset S of $E\Gamma$ let VS denote the vertex-set of the edge-subgraph $\langle S \rangle$. Then a function h: $VS \to [u]$, which coincides with g on $J \cap VS$, may be extended in $u^{|V\Gamma| - |VS \cup J|}$ ways to a function on $V\Gamma$ coinciding with g on J. Let $M(S)$ denote the set of all such h. Reversing the order of the double sum gives

$$R(\Gamma, \alpha) = \sum_{S \subseteq E\Gamma} u^{|V\Gamma| - |VS \cup J|} \sum_{h \in M(S)} \prod_{e \in S} (\hat{h}(e) - 1).$$

Now fix S and consider the term $\sum \Pi(\hat{h}(e) - 1)$. The product is nonzero only if $\hat{h}(e) = 0$ for all e in S, and then it takes the value $(-1)^{|S|}$. But $\hat{h}$ vanishes on all edges of S if and only if h is constant on each component of $\langle S \rangle$. Since h must coincide with the given function g on $J \cap VS$, this means that if two vertices x and y in J lie in the same component of $\langle S \rangle$ then $g(x) = g(y)$ and so $(x, y) \in \alpha$. Consequently, for a nonzero product we must have this condition holding for all vertices of J. In that case, h is determined on the components of $\langle S \rangle$ which meet J, whereas it may take any one of u values on each component of $\langle S \rangle$ which does not meet J. Let $c_J(S)$ denote the number of components of $\langle S \rangle$ not meeting J. The term under consideration is

$$(-1)^{|S|} u^{c_J(S)}$$

if, whenever x and y in J are connected in $\langle S \rangle$ then $(x, y) \in \alpha$, and it is zero otherwise.

THEOREM 3. *The function $R(\Gamma, \alpha)$ can be expanded in terms of the edge-subgraphs $\langle S \rangle$ of Γ, as follows:*

$$R(\Gamma, \alpha) = \sum_{S \in \mathscr{S}(\alpha)} (-1)^{|S|} u^{|V\Gamma| - |VS \cup J| + c_J(S)} \tag{4.1}$$

Here $\mathscr{S}(\alpha)$ denotes the set of subsets S of $E\Gamma$ with the property that whenever x and y in J are connected in $\langle S \rangle$, then $(x, y) \in \alpha$. ∎

We shall need some special cases of Theorem 3. First, take $\alpha = \iota$. Then the set $\mathscr{S} = \mathscr{S}(\iota)$ consists of those subsets S which do not contain a path joining any two distinct members of J. In other words, each component of $\langle S \rangle$ meets J in at most one vertex. So the total number of components of $\langle S \rangle$ is

$$\begin{aligned} c(S) &= c_J(S) + |VS \cap J| \\ &= c_J(S) + |VS| + |J| - |VS \cup J|. \end{aligned}$$

The exponent of u in the formula (4.1) is thus, in this case, equal to $|V\Gamma| - |J| + c(S) - |VS|$. Further,

$$c(S) - |VS| = \epsilon(S) - |S|,$$

where $\epsilon(S)$ is the co-rank (circuit rank) of $\langle S\rangle$. This leads to a formula for $W(\Gamma, \iota) = R(\Gamma, \iota)$;

$$W(\Gamma, \iota) = u^{|V\Gamma|-|J|} \sum_{S\in\mathscr{S}} u^{\epsilon(S)}(-u^{-1})^{|S|}. \tag{4.2}$$

In the application of (4.2) to calculate the entries of a transfer matrix, we shall choose the basic graph Γ in such a way that the contributions to $W(\Gamma, \iota)$ from those subgraphs which contain circuits can be accounted for quite simply. In this situation it is sufficient to begin by calculating the sum

$$\sum_{S\in\mathscr{S}} z^{|S|}$$

and make a correction for the terms with $\epsilon(S) \neq 0$.

5. Approximations

At this point we pause to explain the proposed application of the results in Sections 2, 3, and 4. We shall be dealing with infinite graphs of the kind mentioned in the Introduction: that is, countable, locally finite, vertex-transitive and planar. As examples we cite the three regular plane tessellations and the eight semiregular plane tessellations [4].

An infinite graph of this kind can be regarded as the limit of a double sequence $\{\Psi_{kj}\}$ of finite graphs. For instance, if Ψ is the plane square lattice graph, then Ψ_{kj} is the graph consisting of kj squares arranged in k columns and j rows. For each fixed k we have a family of graphs $(\Psi_k)_j = \Psi_{kj}$ which can be constructed by repeated amalgamation using a graph Λ_k, as in Section 3:

$$(\Psi_k)_j = (\Psi_k)_{j-1} * \Lambda_k \qquad (j \geqslant 1).$$

The number of vertices of Ψ_{kj} is given by an expression of the form

$$V(k, j) = (k + \alpha)(j + \beta) \qquad (\alpha, \beta \text{ constants}).$$

We are interested in finding approximations for the function

$$C_\infty(\Psi, u) = \lim_{k,j\to\infty} \{C(\Psi_{kj}\,;\, u)\}^{1/V(k,j)}$$

We shall proceed in a simple-minded way, since to do more would involve us in theoretical questions about C_∞ which are beyond the scope of the present paper. The formula for C_∞ can be written as

$$\lim_{k\to\infty} [\lim_{j\to\infty} \{C((\Psi_k)_j\,;\, u)\}^{1/(j+\beta)}]^{1/(k+\alpha)}.$$

so that our first task is to find an approximation for the quantity in square brackets.

Now, if $\{\Gamma_j\}$ is a family constructed by repeated amalgamation using the graph Λ, then

$$C(\Gamma_j\,;u) = W(\Lambda, \iota)\, C(\Gamma_{j-1}\,;u) + \text{other terms}.$$

Both sides of this equation are polynomials in u, and the degree of the "other terms" is less than that of the first term. It follows that

$$\lim_{j\to\infty} \{C(\Gamma_j\,;u)\}^{1/j} \sim w(u),$$

where $w = W(\Lambda, \iota)$ and $f(u) \sim g(u)$ means that the limit as $u \to \infty$ of $f(u)/g(u)$ is 1. This gives us the required approximation:

$$\lim_{j\to\infty} C((\Psi_k)_j\,;u)^{1/j+\alpha} \sim w_k(u),$$

where $w_k = W(\Lambda_k, \iota)$.

In the next section we shall derive an approximation of the form

$$w_k(u) \sim u^{f(k)}\, a(u)\{\lambda(u)\}^k,$$

from which it follows that

$$C_\infty(\Psi; u) \sim u^f \lambda(u).$$

Despite the unsophisticated nature of our approximations, it turns out that this is a remarkably good final answer. It has the additional recommendation that it is easy to calculate, as we shall show in Section 6. However, it is clear that the eventual aim of the theory must be to provide exact answers.

6. A Useful Technique

In this section we shall describe a method which will be used to calculate the function $w_k = W(\Lambda_k, \iota)$. We shall follow the plan outlined at the end of Section 4, explaining the method first by reference to a specific example.

Let Λ_k denote the graph of Fig. 2, with $J_1 = \{c_0, c_1, \ldots, c_k\}$ and $J_2 = \{d_0, d_1, \ldots, d_k\}$.

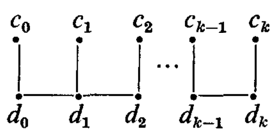


Figure 2.

The construction of Section 3 with $\Lambda = \Lambda_k$ and Γ_0 the path $c_0c_1c_2 \cdots c_k$, gives rise to a family of graphs $\{(\Gamma_k)_j\}$. Our aim is to calculate

$$F_k(z) = \sum_{S \in \mathscr{S}} z^{|S|},$$

where $\mathscr{S}$ is the set of subsets of $E\Lambda_k$ which contain no path joining c_i and c_j $(i \neq j)$.

Each $S C E\Lambda_k$ is partitioned into disjoint subsets $S_0, S_1, \ldots, S_k$ defined by

$$S_0 = S \cap \{c_0, d_0\} \qquad S_i = S \cap \{\{c_i, d_i\}, \{d_{i-1}, d_i\}\} \qquad (1 \leqslant i \leqslant k).$$

Further, associated with each such S is a sequence $(t_0, t_1, \ldots, t_k) = \mathbf{t}(S)$, of zeros and ones, given by the rule

$$t_i = \begin{cases} 1 & \text{if } \langle S \rangle \text{ contains a path } d_id_{i-1} \cdots d_hc_h \ (0 \leqslant h \leqslant i); \\ 0 & \text{otherwise} \end{cases}$$

Our calculation makes use of the fact that the contribution of S_i to $F_k(z)$ depends only on t_{i-1} and t_i. That is, if

$$S_0, S_1, \ldots, S_{i-1}, S_{i+1}, \ldots, S_k$$

are given, then the sum

$$\sum_{S_i} z^{|S_i|}$$

over all S_i for which $S = S_0 \cup \cdots \cup S_i \cup \cdots \cup S_k$ is in $\mathscr{S}$ and $t_{i-1} = \alpha$, $t_i = \beta$, is independent of i $(1 \leqslant i \leqslant k)$. In fact, if we introduce the symbol $q_{\alpha\beta}$ to represent this sum, then the matrix $Q = (q_{\alpha\beta})$ is

$$\begin{bmatrix} z+1 & z^2+z \\ 1 & 2z \end{bmatrix}.$$

It is also convenient to introduce the vector $\mathbf{p} = [1, z]$ to allow for the contribution of S_0. We have

$$\begin{aligned} F_k(z) &= \sum_{S \in \mathscr{S}} z^{|S|} \\ &= \sum_{S \in \mathscr{S}} z^{|S_0|} z^{|S_1|} \cdots z^{|S_k|} \\ &= \sum_{(\alpha, \beta, \gamma, \ldots, \eta, \theta)} \sum z^{|S_0|} z^{|S_1|} \cdots z^{|S_k|}, \end{aligned} \tag{6.1}$$

where the inner sum is over those $S \in \mathscr{S}$ for which $\mathbf{t}(S) = (\alpha, \beta, \gamma, \ldots, \eta, \theta)$. This is the same as the sum over $S_0, \ldots, S_k$ such that $S = S_0 \cup \cdots \cup S_k$ is in $\mathscr{S}$ and $\mathbf{t}(S)$ is as above, which is

$$p_\alpha q_{\alpha\beta} q_{\beta\gamma} \cdots q_{\eta\theta}.$$

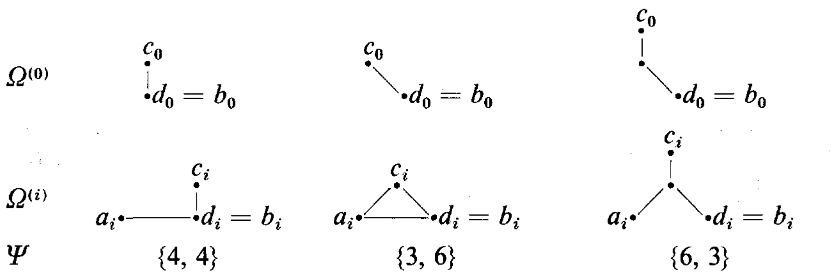
Thus, writing **s** for the column vector $[1, 1]^t$, our result is

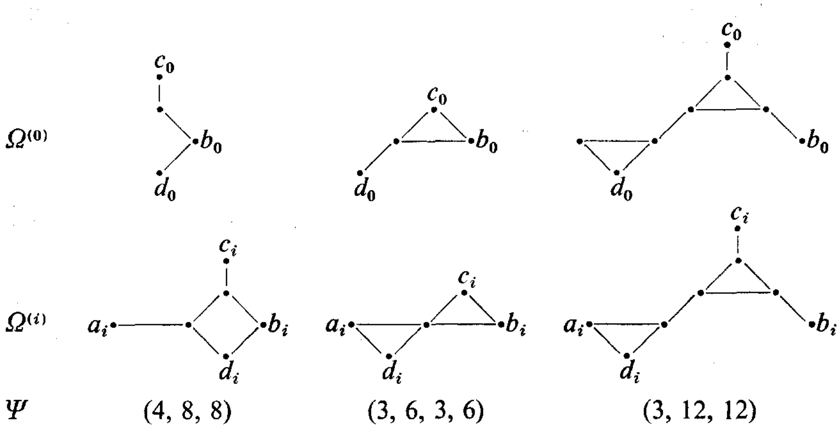
$$F_k(z) = \mathbf{p}Q^k\mathbf{s} = [1 \quad z] \begin{bmatrix} z+1 & z^2+z \\ 1 & 2z \end{bmatrix}^k \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

For instance $F_1(z) = 1 + 3z + 3z^2$, $F_2(z) = 1 + 5z + 10z^2 + 8z^3$, in agreement with examples discussed in Section 3.

The technique just described can be used in the following more general situation. Let Ω denote a graph with distinguished vertices a, b, c, d which are all distinct, except that possibly $b = d$. We begin with a graph $\Omega^{(0)}$ containing vertices b_0, c_0, d_0, and add k copies $\Omega^{(i)}$ of Ω in turn, at each stage identifying the new a_i with the previous b_{i-1} $(1 \leqslant i \leqslant k)$. In the resulting graph Λ_k we have subsets $J_1 = \{c_0, c_1, ..., c_k\}$ and $J_2 = \{d_0, d_1, ..., d_k\}$, as well as distinguished vertices $b_0, b_1, ..., b_k$. It may happen as in our preceding example, that $b_i = d_i$.

In the following table we depict some special choices for $\Omega^{(0)}$ and $\Omega^{(i)}$. The resulting graphs Λ_k can themselves be amalgamated to form families Ψ_{kj}, and in the cases considered these graphs are parts of plane tessellations. The relevant tessellation Ψ in each case is designated by the notation of Fejes Toth [4, p. 43].





In the same way as in the example we may define, for each subset S of $E\Lambda_k$, the subsets $S_i = S \cap E\Omega^{(i)}$ $(0 \leqslant i \leqslant k)$ and the sequence $\mathbf{t}(S)$. The calculation of $F_k(z)$ can then proceed as before, using the matrix Q. However, this method ignores the possibility of circuits in Λ_k; that is, we obtain

$$F_k(z) = \sum_{S\in\mathscr{S}} z^{|S|}$$

rather than the correct expression

$$G_k(z) = \sum_{S\in\mathscr{S}} (-1)^{\epsilon(S)} z^{|S|-\epsilon(S)}$$

(The work of Section 4 shows that the function $w_k = W(\Lambda_k, \iota)$ is

$$u^{|V\Lambda_k - J_i|} G_k(z),$$

where the variable z is just $-1/u$ and u is the basic variable which originally represents the number of colours available.)

In the applications listed above we have constructed Λ_k in such a way that each circuit lies wholly in one of the parts $\Omega^{(i)}$ $(0 \leqslant i \leqslant k)$. Then the correct expression replacing (6.1) is

$$\sum_{(\alpha,\beta,\gamma,\ldots,\eta,\theta)} \sum (-1)^{\epsilon(S_0)} z^{|S_0|-\epsilon(S_0)} (-1)^{\epsilon(S_1)} z^{|S_1|-\epsilon(S_1)} \cdots (-1)^{\epsilon(S_k)} z^{|S_k|-\epsilon(S_k)}.$$

The necessary changes can be made merely by altering the definition of Q. That is, we introduce a corrected 2×2 matrix C whose entries $c_{\alpha\beta}$ are the sums.

$$\sum_{S_i} (-1)^{\epsilon(S_i)} z^{|S_i|-\epsilon(S_i)},$$

over all S_i for which $S_0 \cup \cdots \cup S_i \cup \cdots \cup S_k = S$ is in $\mathscr{S}$ and $t_{i-1} = \alpha$, $t_i = \beta$.

For example, in the case of the triangular tesellation $\{3, 6\}$ we have

$$Q = \begin{bmatrix} 2z+1 & z^3+3z^2+z \\ 1 & 2z \end{bmatrix} \qquad C = \begin{bmatrix} 2z+1 & 2z^2+z \\ 1 & 2z \end{bmatrix}.$$

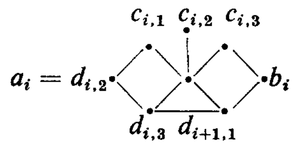
Here the correction corresponds to the case when all three edges of $\Omega^{(i)}$ are in S_i. This applies (for $S \in \mathscr{S}$) only when $t_{i-1} = 0$ and $t_i = 1$, and the correction amounts to replacing a term z^3 by $(-1)\, z^{3-1} = -z^2$.

It is possible that the vector $\mathbf{p}$ corresponding to $\Omega^{(0)}$ may also require correction. Some corrected $\mathbf{p}$ vectors and C matrices are as follows:

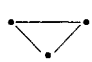
p	$[2z^2 + 3z + 1 \quad z^3 + z^2]$	$[2z^2 + 3z + 1 \quad 2z^3 + 3z^2 + z]$
C	$\begin{bmatrix} 6z^4 + 15z^3 + 14z^2 + 6z + 1 & 3z^5 + 6z^4 + 4z^3 + z^2 \\ 12z^3 + 14z^2 + 6z + 1 & 9z^4 + 6z^3 + z^2 \end{bmatrix}$	$\begin{bmatrix} 4z^3 + 8z^2 + 5z + 1 & 4z^4 + 8z^3 + 5z^2 + z \\ 6z^2 + 5z + 1 & 8z^3 + 6z^2 + z \end{bmatrix}$
Ψ	(4, 8, 8)	(3, 6, 3, 6)

Note that this method may be generalized to allow Ω to contain more than one vertex of type c, provided it has the same number as of type d. It can then be used on the remaining semiregular tessellations, e.g., for $\Psi = (3, 3, 4, 3, 4)$:

$\Omega^{(0)}$: 

$\Omega^{(i)}$: 

We may also split Ω itself into subgraphs each of which has one type a and one type b vertex, but not necessarily equal numbers of types c and d. For example, for $\Psi = (3, 6, 3, 6)$ we can consider

 and 

separately to obtain

$$C = \begin{bmatrix} 2z^2 + 3z + 1 & 0 \\ 2z + 1 & 2z^2 + z \end{bmatrix} \begin{bmatrix} 2z + 1 & 2z^2 + z \\ 1 & 2z \end{bmatrix}.$$

We have now obtained an exact expression for $w_k = W(\Lambda_k, \iota)$ in terms of the variable $z = -1/u$:

$$w_k = u^{f(k)} \cdot \mathbf{p}C^k\mathbf{s} \quad (f(k) = | V\Lambda_k - J |).$$

Our aim is to get an approximation for the limit as $k \to \infty$ of $w_k^{1/k}$, and for this we proceed as follows. The matrix C may be expressed as

$$C = \lambda A + \mu B,$$

where λ, μ are the eigenvalues of C, and A, B are mutually orthogonal idempotents. Writing $\mathbf{p}A\mathbf{s} = a$, $\mathbf{p}B\mathbf{s} = b$, we get

$$w_k = u^{f(k)}(a\lambda^k + b\mu^k).$$

Now $f(k)$ is a linear function of k, so $f(k)/k$ tends to a constant f as $k \to \infty$. Thus, we have the approximation foreshadowed in Section 5,

$$C_\infty(\Psi; u) \sim u^f \lambda(u),$$

where λ is assumed to be the dominant eigenvalue of C as $u \to \infty$.

In the case of the square tessellation $\Psi = \{4, 4\}$ we get $f = 1$ and λ is the dominant root of the equation

$$\lambda^2 - (3z + 1)\,\lambda + z(z + 1) = 0.$$

Consequently

$$C_\infty(\Psi; u) \sim \tfrac{1}{2}[u - 3 + \sqrt{(u^2 - 2u + 5)}].$$

This is comparable with the approximation

$$C_\infty(\Psi; u) \sim u(1 - u^{-1})^2 \left[1 + \frac{1}{(u-1)^3}\right]\left[1 + \frac{1}{(u-1)^7}\right]$$

obtained by Baker [1] by means of a multiplicative expansion [2, p. 84]. Similar approximations can be obtained for the other graphs whose C matrices are listed above.

We close by remarking that the justification for this paper is not the useful approximations of the last two sections, however fortunately accurate they may be. Rather, it is the hope that the theoretical results of Sections 2, 3, and 4 may lead to exact results for infinite graphs.

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